Accurate Evaluation of Fermi-Dirac Integrals and Their Derivatives for Arbitrary Degeneracy and Relativity (astro-ph/9509124)

Juan Antonio Miralles[1]

Departamento de Astronomía y Astrofísica, Universitat de València,

Dr. Moliner 50, 46100–Burjassot (Valencia), SPAIN

and

Kenneth A. Van Riper

Transport Methods Group, Los Alamos National Laboratory

MS B226, Los Alamos, NM 87545, USA

Received ______________________; accepted ______________________

---

[1]IGPP, Los Alamos National Laboratory

ABSTRACT

The equation of state of an ideal Fermi gas is expressed in terms of Fermi-Dirac integrals. We give formulae for evaluation the Fermi-Dirac integrals of orders $^1/_2$, $^3/_2$, and $^5/_2$ and their derivatives in various limits of non- and extreme degeneracy and relativity. We provide tables and a Fortran subroutine for numerical evaluation of the integrals and derivatives when a limit does not apply. The functions can be evaluated to better than 1% accuracy for any temperature and density using these methods.

Subject headings: Equation of State: electrons, Fermi gas

1. Introduction

The electron constituent of neutron star envelopes exists in various stages of degeneracy, from a classical gas on the surface to an extremely degenerate Fermi gas in the interior. Convective regimes may exist within the envelope (Urpin 1981). Whether convection does occur depends on the magnitude of the temperature gradient compared to the adiabatic gradient, which is most conveniently calculated from the adiabatic index

$$\Gamma_2 = \frac{1}{1 - 1/G} \tag{1-1}$$

where

$$G = \frac{T}{P}\frac{\partial P}{\partial T}\bigg|_\rho + \frac{\rho}{P}\frac{\partial P}{\partial \rho}\bigg|_T \left[\left(\frac{\partial E}{\partial \rho}\bigg|_T\right) \Big/ \left(\frac{\partial P}{\partial \rho}\bigg|_T\right)\right] \tag{1-2}$$

An estimate of the role of convection requires accurate values for the temperature T and density ρ derivatives of the Pressure P and energy density E. The envelope constituents are electrons, ions, and radiation. The total equation of state, which we treat elsewhere, also contains contributions from the Coulomb correction to the ions and takes into account the changes in binding energy and electron density due to the variation of ionization level with T and ρ.

The electrons make a significant contribution to the equation of state (EOS) in the nondegenerate regime, and dominate when they are degenerate. We treat the electrons as an ideal Fermi gas of arbitrary degeneracy and relativity. We have previously evaluated the nonderivative electron EOS by a scheme which interpolates in a two-dimensional table of Fermi-Dirac integrals (eq. [2-7] below) or uses asymptotic formulae in degenerate, nondegenerate, relativistic, and nonrelativistic limits, where applicable. Those limiting formulae are described in Bludman & Van Riper (1977).

The evaluation scheme for the direct function is not sufficiently accurate for the

calculation of the EOS derivatives by direct numerical differences. Such a scheme becomes ever more hopeless as the degeneracy increases. The fundamental problem is obtaining temperature derivatives of quantities which have a vanishingly small temperature dependence. This problem also presented itself in our derivation of asymptotic formulae for the degenerate T derivatives. Since Urpin (1981) had suggested the possibility of convection in the degenerate regime, we were particularly interested in obtaining an accurate adiabatic index there.

Our method of evaluating the derivatives of the EOS relies on the derivatives of the Fermi-Dirac integrals, which are found by a scheme exactly analogous to our method for the integrals themselves. We prepare a table of the derivatives by accurate numerical integration for intermediate degeneracy and relativity and derived formulae (sometimes relying on other tabulated functions) in the various limits. The interpolation can be made with either a fast second order method or with more accurate polynomial schemes.

We present those formulae here. We will also publish our tables of the Fermi-Dirac functions, their derivatives, modified Bessel functions of the first and second kind, and some Fermi integrals on the Astrophysical Journal CDROM. We will also include FORTRAN subroutines in which our method is implemented. This paper serves as documentation for both tables and subroutines. As such, we will give a brief review of the casting of the EOS in terms of the Fermi-Dirac functions and will also show the limiting formulae for the functions along with the corresponding formulae for their derivatives.

The monikers "Fermi-Dirac" and "Fermi" receive no consistent usage in the literature. We reserve the term *Fermi-Dirac* integral (or function) for the bivariate (temperature and degeneracy parameter η) functions defined in equation (2-7). In the relativistic and nonrelativistic limits, these functions reduce to expressions involving integrals (eq. [4-1] below) which depend on η alone; we refer to these latter as integrals as the *Fermi* functions.

Numerous studies of the Fermi-Dirac integrals and methods for their rapid evaluation exist in the literature (though few of these explicitly treat the derivatives). An excellent general reference is chapter 24 of the book of Cox & Giuli (1968), which describes the general theory, formulae for many limiting cases, and tabulations of the Fermi-Dirac integrals (based on unpublished work by Terry W. Edwards). The literature contains a number of schemes for calculating one or more of the integrals. Kippenhahn & Thomas (1964; see also Kippenhahn, et al. 1967) give a power series expansion for the evaluation of the thermodynamic quantities n, P, and E which is valid for non- and mildly degenerate gases. Divine (1965) gives a method, based on third order rational function approximations to Fermi integrals of orders $^1/_2$, $^3/_2$, 2, $^5/_2$, and 3, which gives n, P, and E to a stated accuracy of 0.3% for arbitrary degrees of degeneracy and relativity. Guess (1966) considers an set of functions (Q_n) equivalent to the Fermi-Dirac functions, and gives series expansions for high and low temperatures and degeneracies, accompanied by a tables for the central regime where none of those limits apply. Tooper (1969), considering relativistic gases for arbitrary degeneracy, derives series expansions in the non- and extremely-degenerate limits and discusses methods for numerical integration in the intermediate regime. Beaudet & Tassoul (1971) give simple formulae with which n, P, and E can be evaluated to an accuracy of several percent for the relativistic and/or degenerate regimes. Bludman & Van Riper (1977) give simple formulae, accurate to 0.5% for the semi-degenerate, relativistic and nonrelativistic regimes. Nadyozhin (1974) and Blinnikov & Rudzskii (1988) consider the limiting cases of extreme relativity and give a number of series expansions. Eggleton, et al. (1973) derived fitting formulae for the n, P, and E of an ideal electron gas for a range of T and ρ covering the regime of arbitrary degeneracy and relativity. Evaluation of the thermodynamic quantities with $5^{\rm th}$ order formulae (their Table 4) agree with values based on our main table numerical integrations to 0.04%. An extension of a $4^{\rm th}$ order method of Eggleton, et al. given by Pols et al. (1995, Appendix A), while thermodynamically

consistent, agrees with our integrations to 0.3%.

There is a considerable literature dealing with the Fermi integrals. We do not mention any of this work here, except to take note of Antia's (1993) rational function approximations for Fermi integrals of several half-integral orders with stated maximum relative error of 10^{-12}.

In the next § we give the formulae for several thermodynamic quantities of an ideal Fermi gas in terms of the Fermi-Dirac integrals. Asymptotic formulae in the degenerate limit are presented in § 3, including special treatment necessary for the temperature derivatives of the pressure and energy density. The treatments in other asymptotic regions are discussed in §4. Section 5 covers the numerical details, including the integration and interpolation methods and the accuracy and efficiency of the scheme for various interpolation orders.

2. Thermodynamics of an Ideal Fermi Gas

We will work throughout in terms of the dimensionless degeneracy and temperature

$$\eta = \frac{\mu}{k_{\rm B}T}, \quad \text{and} \quad \beta = \frac{k_{\rm B}T}{mc^2}, \tag{2-1}$$

where μ is the chemical potential and m is the mass of the fermion (this theory is also applicable to ideal neutron and proton gases). Constants such as the Boltzman constant $k_{\rm B}$ have their usual meaning throughout this paper. The gas is degenerate (nondegenerate) for $\eta \gg 0$ ($\eta \ll 0$). We will refer to relativity regimes based on the value of β, with the gas being relativistic (nonrelativistic) for $\beta \gg 1$ ($\beta \ll 1$). The gas also becomes relativistic at high density when the $\mu \gg mc^2$ or, equivalently, when $\eta\beta \gg 1$. In practice, we take the degenerate (nondegenerate) regime to be $\eta \geq 70$ ($\eta \leq -30$) and the relativistic (nonrelativistic) regime to be $\beta \geq 10^4$ ($\beta \leq 10^{-6}$).

The zero of energy for the particles is chosen so that the thermodynamic potential is

$$\Omega = -Vk_{\mathrm{B}}T \int \frac{gd^3p}{h^3} \ln \left[\frac{1 + \exp\left(\mu - \epsilon\right)}{k_{\mathrm{B}}T} \right], \tag{2-2}$$

where p is the momentum, g is the statistical weight, and

$$\epsilon = \sqrt{(mc^2)^2 + (pc)^2} - mc^2 \tag{2-3}$$

is the kinetic energy. With the energy so defined, μ does not contain the rest mass. (We do not consider antiparticles—positrons—in this work; neutron star envelopes do not encounter the high T and low ρ where e^+ appear in significant numbers.)

The number density n, pressure P, and energy density (per volume) E of an ideal Fermi gas are

$$n = \frac{8\pi\sqrt{2}m^3c^3}{h^3}\beta^{\frac{3}{2}} \left[F_{1/2}(\eta,\beta) + \beta F_{3/2}(\eta,\beta)\right], \tag{2-4}$$

$$P = \frac{16\pi\sqrt{2}m^4c^5}{3h^3}\beta^{\frac{5}{2}} \left[F_{3/2}(\eta,\beta) + \frac{1}{2}\beta F_{5/2}(\eta,\beta)\right], \tag{2-5}$$

and

$$E = \frac{8\pi\sqrt{2}m^4c^5}{h^3}\beta^{\frac{5}{2}} \left[F_{3/2}(\eta,\beta) + \beta F_{5/2}(\eta,\beta)\right], \tag{2-6}$$

where the Fermi-Dirac integral of order k is defined as

$$F_k(\eta,\beta) = \int_0^\infty \frac{x^k \left(1 + \frac{1}{2}\beta x\right)^{\frac{1}{2}}}{\exp(x-\eta)+1}\, dx. \tag{2-7}$$

The energy and pressure derivatives with respect to T and n are

$$\left.\frac{\partial P}{\partial n}\right|_T = \frac{2}{3}mc^2\beta \left(\frac{\partial F_{3/2}}{\partial \eta} + \frac{1}{2}\beta\frac{\partial F_{5/2}}{\partial \eta}\right) \Big/ \left(\frac{\partial F_{1/2}}{\partial \eta} + \beta\frac{\partial F_{3/2}}{\partial \eta}\right), \tag{2-8}$$

$$\left.\frac{\partial E}{\partial n}\right|_T = mc^2\beta \left(\frac{\partial F_{3/2}}{\partial \eta} + \beta\frac{\partial F_{5/2}}{\partial \eta}\right) \Big/ \left(\frac{\partial F_{1/2}}{\partial \eta} + \beta\frac{\partial F_{3/2}}{\partial \eta}\right), \tag{2-9}$$

$$\left.\frac{\partial P}{\partial T}\right|_n = \frac{16\pi\sqrt{2}m^3c^3}{3h^3}k_{\rm B}\beta^{\frac{3}{2}}\left\{\frac{5}{2}F_{3/2} + \beta\frac{\partial F_{3/2}}{\partial\beta} + \frac{7}{4}\beta F_{5/2} + \frac{1}{2}\beta^2\frac{\partial F_{5/2}}{\partial\beta}\right.$$
$$\left. - \frac{\left(\frac{\partial F_{3/2}}{\partial\eta} + \frac{1}{2}\beta\frac{\partial F_{5/2}}{\partial\eta}\right)\left(\frac{3}{2}F_{1/2} + \beta\frac{\partial F_{1/2}}{\partial\beta} + \frac{5}{2}\beta F_{3/2} + \beta^2\frac{\partial F_{3/2}}{\partial\beta}\right)}{\left(\frac{\partial F_{1/2}}{\partial\eta} + \beta\frac{\partial F_{3/2}}{\partial\eta}\right)}\right\}, \tag{2-10}$$

and

$$\left.\frac{\partial E}{\partial T}\right|_n = \frac{8\pi\sqrt{2}m^3c^3}{h^3}k_{\rm B}\beta^{\frac{3}{2}}\left\{\frac{5}{2}F_{3/2} + \beta\frac{\partial F_{3/2}}{\partial\beta} + \frac{7}{2}\beta F_{5/2} + \beta^2\frac{\partial F_{5/2}}{\partial\beta}\right.$$
$$\left. - \frac{\left(\frac{\partial F_{3/2}}{\partial\eta} + \beta\frac{\partial F_{5/2}}{\partial\eta}\right)\left(\frac{3}{2}F_{1/2} + \beta\frac{\partial F_{1/2}}{\partial\beta} + \frac{5}{2}\beta F_{3/2} + \beta^2\frac{\partial F_{3/2}}{\partial\beta}\right)}{\left(\frac{\partial F_{1/2}}{\partial\eta} + \beta\frac{\partial F_{3/2}}{\partial\eta}\right)}\right\}, \tag{2-11}$$

where the derivatives of the Fermi-Dirac integrals are

$$\frac{\partial F_k}{\partial\eta}(\eta,\beta) = \int_0^\infty \frac{x^k\left(1+\frac{1}{2}\beta x\right)^{\frac{1}{2}}}{\left[\exp\left(\frac{x-\eta}{2}\right) + \exp\left(\frac{\eta-x}{2}\right)\right]^2}\,dx \tag{2-12}$$

and

$$\frac{\partial F_k}{\partial\beta}(\eta,\beta) = \int_0^\infty \frac{x^{(k+1)}\left(1+\frac{1}{2}\beta x\right)^{-\frac{1}{2}}}{4\left[\exp(x-\eta)+1\right]}\,dx. \tag{2-13}$$

3. The Degenerate Limit

3.1. Temperature Derivatives

The temperature dependence of P and E in the degenerate regime is vanishingly small; obtaining accurate temperature derivatives is accordingly problematic. In particular, the computer representation of temperature-independent terms in (1) and (2) lacks sufficient resolution to ensure cancellations which should occur. We implement these cancellations analytically and use the resulting *thermal* terms in the degenerate temperature derivatives.

When $\eta > 70$, the following are used instead of (2-10) and (2-11):

$$\left.\frac{\partial P}{\partial T}\right|_{n,\mathrm{ED}} = \frac{16\pi\sqrt{2}m^3c^3}{3h^3}k_{\mathrm{B}}\beta^{\frac{3}{2}}\left(T_1 - T_2 + T_3 - T_4\right) \Big/ \left(\frac{\partial F_{1/2}}{\partial \eta} + \beta\frac{\partial F_{3/2}}{\partial \eta}\right) \tag{3-1a}$$

where the terms in the numerator are

$$\begin{aligned}
T_1 &= \left(\frac{5}{2}F_{3/2} + \beta\frac{\partial F_{3/2}}{\partial \beta} + \frac{7}{4}\beta F_{5/2} + \frac{\beta^2}{2}\frac{\partial F_{5/2}}{\partial \beta}\right)\left(\frac{\partial F_{1/2}^{\mathrm{th}}}{\partial \eta} + \beta\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta}\right), \\
T_2 &= \left(\frac{3}{2}F_{1/2} + \beta\frac{\partial F_{1/2}}{\partial \beta} + \frac{5}{2}\beta F_{3/2} + \beta^2\frac{\partial F_{3/2}}{\partial \beta}\right)\left(\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta} + \frac{1}{2}\beta\frac{\partial F_{5/2}^{\mathrm{th}}}{\partial \eta}\right), \\
T_3 &= \left(\frac{\partial F_{1/2}}{\partial \eta} + \beta\frac{\partial F_{3/2}}{\partial \eta} - \frac{\partial F_{1/2}^{\mathrm{th}}}{\partial \eta} - \beta\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta}\right)\left(\frac{5}{2}F_{3/2}^{\mathrm{th}} + \beta\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \beta} + \frac{7}{4}\beta F_{5/2}^{\mathrm{th}} + \frac{\beta^2}{2}\frac{\partial F_{5/2}^{\mathrm{th}}}{\partial \beta}\right), \\
T_4 &= \left(\frac{\partial F_{3/2}}{\partial \eta} + \frac{1}{2}\beta\frac{\partial F_{5/2}}{\partial \eta} - \frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta} - \frac{1}{2}\beta\frac{\partial F_{5/2}^{\mathrm{th}}}{\partial \eta}\right)\left(\frac{3}{2}F_{1/2}^{\mathrm{th}} + \beta\frac{\partial F_{1/2}^{\mathrm{th}}}{\partial \beta} + \frac{5}{2}\beta F_{3/2}^{\mathrm{th}} + \beta^2\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \beta}\right),
\end{aligned} \tag{3-1b}$$

and

$$\left.\frac{\partial E}{\partial T}\right|_{n,\mathrm{ED}} = \frac{8\pi\sqrt{2}m^3c^3}{3h^3}k_{\mathrm{B}}\beta^{\frac{3}{2}}\left(T_5 - T_6 + T_7 - T_8\right) \Big/ \left(\frac{\partial F_{1/2}}{\partial \eta} + \beta\frac{\partial F_{3/2}}{\partial \eta}\right), \tag{3-2a}$$

where

$$\begin{aligned}
T_5 &= \left(\frac{5}{2}F_{3/2} + \beta\frac{\partial F_{3/2}}{\partial \beta} + \frac{7}{2}\beta F_{5/2} + \beta^2\frac{\partial F_{5/2}}{\partial \beta}\right)\left(\frac{\partial F_{1/2}^{\mathrm{th}}}{\partial \eta} + \beta\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta}\right), \\
T_6 &= \left(\frac{3}{2}F_{1/2} + \beta\frac{\partial F_{1/2}}{\partial \beta} + \frac{5}{2}\beta F_{3/2} + \beta^2\frac{\partial F_{3/2}}{\partial \beta}\right)\left(\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta} + \beta\frac{\partial F_{5/2}^{\mathrm{th}}}{\partial \eta}\right), \\
T_7 &= \left(\frac{\partial F_{1/2}}{\partial \eta} + \beta\frac{\partial F_{3/2}}{\partial \eta} - \frac{\partial F_{1/2}^{\mathrm{th}}}{\partial \eta} - \beta\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta}\right)\left(\frac{5}{2}F_{3/2}^{\mathrm{th}} + \beta\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \beta} + \frac{7}{2}\beta F_{5/2}^{\mathrm{th}} + \beta^2\frac{\partial F_{5/2}^{\mathrm{th}}}{\partial \beta}\right),
\end{aligned}$$

and

$$T_8 = \left(\frac{\partial F_{3/2}}{\partial \eta} + \beta\frac{\partial F_{5/2}}{\partial \eta} - \frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \eta} - \beta\frac{\partial F_{5/2}^{\mathrm{th}}}{\partial \eta}\right)\left(\frac{3}{2}F_{1/2}^{\mathrm{th}} + \beta\frac{\partial F_{1/2}^{\mathrm{th}}}{\partial \beta} + \frac{5}{2}\beta F_{3/2}^{\mathrm{th}} + \beta^2\frac{\partial F_{3/2}^{\mathrm{th}}}{\partial \beta}\right). \tag{3-2b}$$

3.2. Degenerate Fermi-Dirac Functions

The asymptotic degenerate formulae are given in terms of

$$y = \sqrt{A^2 - 1} \tag{3-3}$$

where

$$A = 1 + \eta\beta \tag{3-4}$$

When the degenerate gas is relativistic, $\eta\beta \gg 1$ and $y \approx \eta\beta \gg 1$. Similarly, in the nonrelativistic limit, $\eta\beta \ll 1$, $y \approx \sqrt{2\eta\beta} << 1$. For most functions, different formulae are used depending the value of y.

The following expressions for the Fermi-Dirac functions and the thermal terms are only used in the extreme degenerate limit ($\eta > 70$). In that limit, the first few terms in the degenerate expansions are sufficient (additional terms in the expansion may be found in Cox & Giuli [1968, chapter 24]). The expressions are

$$F_{1/2}(\eta,\beta) \simeq \frac{1}{\sqrt{2}}\beta^{-\frac{3}{2}} f_{1/2} + \frac{\pi^2}{6\sqrt{2}}\eta^{-\frac{1}{2}}\frac{1+\eta\beta}{\sqrt{2+\eta\beta}} \tag{3-5a}$$

$$F^{\mathrm{th}}_{1/2}(\eta,\beta) = \frac{\pi^2}{6\sqrt{2}}\eta^{-\frac{1}{2}}\frac{1+\eta\beta}{\sqrt{2+\eta\beta}} \tag{3-5b}$$

$$F_{3/2}(\eta,\beta) \simeq \frac{1}{\sqrt{2}}\beta^{-\frac{5}{2}} f_{3/2} + \frac{\pi^2}{6\sqrt{2}}\eta^{\frac{1}{2}}\frac{3+2\eta\beta}{\sqrt{2+\eta\beta}}, \tag{3-6a}$$

$$F^{\mathrm{th}}_{3/2}(\eta,\beta) = \frac{\pi^2}{6\sqrt{2}}\eta^{\frac{1}{2}}\frac{3+2\eta\beta}{\sqrt{2+\eta\beta}}, \tag{3-6b}$$

and

$$F_{5/2}(\eta,\beta) \simeq \frac{1}{\sqrt{2}}\beta^{-\frac{7}{2}} f_{5/2} + \frac{\pi^2}{6\sqrt{2}}\eta^{\frac{3}{2}}\frac{5+3\eta\beta}{\sqrt{2+\eta\beta}}, \tag{3-7a}$$

$$F^{\mathrm{th}}_{5/2}(\eta,\beta) = \frac{\pi^2}{6\sqrt{2}}\eta^{\frac{3}{2}}\frac{5+3\eta\beta}{\sqrt{2+\eta\beta}}, \tag{3-7b}$$

where

$$f_{1/2} = \begin{cases} \frac{1}{2}\left[yA - \ln(y+A)\right] & \text{if } y > 0.05 \\ \frac{1}{3}y^3 - \frac{1}{10}y^5 + \frac{3}{56}y^7 - \frac{5}{144}y^9 + \frac{35}{1408}y^{11} & \text{if } y \le 0.05 \end{cases}, \tag{3-8}$$

$$f_{3/2} = \begin{cases} \frac{1}{3}y^3 - \frac{1}{2}\left[yA - \ln(y+A)\right] & \text{if } y > 0.05 \\ \frac{1}{10}y^5 - \frac{3}{56}y^7 + \frac{5}{144}y^9 - \frac{35}{1408}y^{11} & \text{if } y \le 0.05 \end{cases}, \tag{3-9}$$

and

$$f_{5/2} = \begin{cases} \frac{5}{8}yA\left(1 + \frac{2}{5}y^2\right) - \frac{2}{3}y^3 - \frac{5}{8}\ln(y+A) & \text{if } y > 0.1 \\ \frac{1}{28}y^7 - \frac{1}{36}y^9 + \frac{15}{704}y^{11} & \text{if } y \le 0.1 \end{cases}. \tag{3-10}$$

The small y expansions are used to avoid loss of accuracy due to strong cancellations in nonrelativistic limit.

3.3. Degenerate η-Derivatives

The η derivatives and the corresponding thermal functions are, for all values of the relativity parameter y,

$$\frac{\partial F_{1/2}(\eta,\beta)}{\partial \eta} \simeq \frac{1}{\sqrt{2}}\eta^{\frac{1}{2}}(2+\eta\beta)^{\frac{1}{2}}\left[1 - \frac{\pi^2}{6}\,\frac{1}{\eta^2(2+\eta\beta)^2}\right], \tag{3-11a}$$

$$\frac{\partial F^{\rm th}_{1/2}(\eta,\beta)}{\partial \eta} = -\frac{\pi^2}{6\sqrt{2}}\,\frac{1}{\eta^{\frac{3}{2}}(2+\eta\beta)^{\frac{3}{2}}}, \tag{3-11b}$$

$$\frac{\partial F_{3/2}(\eta,\beta)}{\partial \eta} \simeq \frac{1}{\sqrt{2}}\eta^{\frac{3}{2}}(2+\eta\beta)^{\frac{1}{2}}\left[1 + \frac{\pi^2}{6}\,\frac{3+6\eta\beta+2\eta^2\beta^2}{\eta^2(2+\eta\beta)^2}\right], \tag{3-12a}$$

$$\frac{\partial F^{\rm th}_{3/2}(\eta,\beta)}{\partial \eta} = \frac{\pi^2}{6\sqrt{2}}\,\frac{3+6\eta\beta+2\eta^2\beta^2}{\eta^{\frac{1}{2}}(2+\eta\beta)^{\frac{3}{2}}}, \tag{3-12b}$$

$$\frac{\partial F_{5/2}(\eta,\beta)}{\partial \eta} \simeq \frac{1}{\sqrt{2}}\eta^{\frac{5}{2}}(2+\eta\beta)^{\frac{1}{2}}\left[1 + \frac{\pi^2}{6}\,\frac{15+20\eta\beta+6\eta^2\beta^2}{\eta^2(2+\eta\beta)^2}\right], \tag{3-13a}$$

and

$$\frac{\partial F^{\rm th}_{5/2}(\eta,\beta)}{\partial \eta} = \frac{\pi^2}{6\sqrt{2}}\eta^{\frac{1}{2}}\,\frac{15+20\eta\beta+6\eta^2\beta^2}{(2+\eta\beta)^{\frac{3}{2}}}. \quad \text{(3-13b)}$$

3.4. Degenerate β-Derivatives

The β derivatives and accompanying thermal terms are

$$\frac{\partial F_{1/2}(\eta,\beta)}{\partial \beta} \simeq \frac{\partial F_{1/2}(\eta,\beta)}{\partial \eta}\frac{\eta}{\beta} - \frac{3}{2}\frac{1}{\sqrt{2}}\beta^{-\frac{5}{2}}f_{1/2} + \frac{1}{2}\frac{\pi^2}{6\sqrt{2}}\,\frac{1}{\eta^{\frac{1}{2}}\beta}\frac{1+\eta\beta}{(2+\eta\beta)^{\frac{1}{2}}}, \quad \text{(3-14a)}$$

$$\frac{\partial F^{\rm th}_{1/2}(\eta,\beta)}{\partial \beta} = \frac{\partial F^{\rm th}_{1/2}(\eta,\beta)}{\partial \eta}\frac{\eta}{\beta} + \frac{1}{2}\frac{\pi^2}{6\sqrt{2}}\,\frac{1}{\eta^{\frac{1}{2}}\beta}\frac{1+\eta\beta}{(2+\eta\beta)^{\frac{1}{2}}}, \quad \text{(3-14b)}$$

$$\frac{\partial F_{3/2}(\eta,\beta)}{\partial \beta} \simeq \frac{\partial F_{3/2}(\eta,\beta)}{\partial \eta}\frac{\eta}{\beta} - \frac{5}{2}\frac{1}{\sqrt{2}}\beta^{-\frac{7}{2}}f_{3/2} - \frac{1}{2}\frac{\pi^2}{6\sqrt{2}}\,\frac{\eta^{\frac{1}{2}}}{\beta}\frac{3+2\eta\beta}{(2+\eta\beta)^{\frac{1}{2}}}, \quad \text{(3-15a)}$$

$$\frac{\partial F^{\rm th}_{3/2}(\eta,\beta)}{\partial \beta} = \frac{\partial F^{\rm th}_{3/2}(\eta,\beta)}{\partial \eta}\frac{\eta}{\beta} - \frac{1}{2}\frac{\pi^2}{6\sqrt{2}}\,\frac{\eta^{\frac{1}{2}}}{\beta}\frac{3+2\eta\beta}{(2+\eta\beta)^{\frac{1}{2}}}, \quad \text{(3-15b)}$$

$$\frac{\partial F_{5/2}(\eta,\beta)}{\partial \beta} \simeq \frac{\partial F_{5/2}(\eta,\beta)}{\partial \eta}\frac{\eta}{\beta} - \frac{7}{2}\frac{1}{\sqrt{2}}\beta^{-\frac{9}{2}}f_{5/2} - \frac{3}{2}\frac{\pi^2}{6\sqrt{2}}\,\frac{\eta^{\frac{3}{2}}}{\beta}\frac{5+3\eta\beta}{(2+\eta\beta)^{\frac{1}{2}}}, \quad \text{(3-16a)}$$

and

$$\frac{\partial F^{\rm th}_{3/2}(\eta,\beta)}{\partial \beta} = \frac{\partial F^{\rm th}_{5/2}(\eta,\beta)}{\partial \eta}\frac{\eta}{\beta} - \frac{3}{2}\frac{\pi^2}{6\sqrt{2}}\,\frac{\eta^{\frac{3}{2}}}{\beta}\frac{5+3\eta\beta}{(2+\eta\beta)^{\frac{1}{2}}}. \quad \text{(3-16b)}$$

4. Asymptotic Limits when Not Extremely Degenerate

4.1. Arbitrary Degeneracy

In the relativistic ($\beta \gg 1$) and nonrelativistic ($\beta \ll 1$) limits the Fermi-Dirac functions can be expressed in terms of the simpler Fermi functions G, which only depend on η. The Fermi integrals are given by

$$G_k(\eta) = \int_0^\infty \frac{x^k}{\exp(x-\eta)+1}\, dx \tag{4-1}$$

with derivatives

$$\frac{\partial G_k}{\partial \eta} = \int_0^\infty \frac{x^k}{\left\{\exp\left[\frac{(x-\eta)}{2}\right] + \exp\left[\frac{(\eta-x)}{2}\right]\right\}^2}\, dx. \tag{4-2}$$

We require these functions for orders $k = {}^1\!/_2,\ 1,\ {}^3\!/_2,\ 2,\ {}^5\!/_2,\ 3,$ and ${}^7\!/_2$. We have prepared, by numerical integration, tables of G and $\partial G/\partial \eta$, for each of those 7 orders, on a grid of integral values of $-30 \le \eta \le 70$. The evaluation of G and $\partial G/\partial \eta$ in the following formulae is accomplished by interpolation. We discuss the tables in more detail below.

4.1.1. Arbitrary Degeneracy and NonRelativistic

In the nonrelativistic limit, $\beta < 10^{-6}$, the Fermi-Dirac functions and their η-derivatives reduce to the Fermi functions and their derivatives:

$$F_k(\eta,\beta) \simeq G_k(\eta), \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2 \tag{4-3}$$

and

$$\frac{\partial F_k(\eta,\beta)}{\partial \eta} \simeq \frac{\partial G_k(\eta)}{\partial \eta}, \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2. \tag{4-4}$$

The β-derivatives

$$\frac{\partial F_k(\eta,\beta)}{\partial \beta} \simeq \frac{1}{4} G_{k+1}(\eta), \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2. \tag{4-5}$$

involve the next higher order Fermi function. The highest order derivative $\propto G_{7/2}$ only appears in the expressions for the β-derivatives of P and E, multiplied by $\beta << 1$. Because

$G_{7/2}$ is not otherwise required, we carry the $G_{7/2}$ table in a separate file for the convenience of implementations where setting $\partial F_{5/2}(\eta,\beta)/\partial\beta = 0$ in the nonrelativistic limit is sufficient.

4.1.2. Arbitrary Degeneracy and Extremely Relativistic

In the relativistic limit, $\beta > 10^4$,

$$F_k(\eta,\beta) = 2\beta\left[\frac{\partial F_k(\eta,\beta)}{\partial\beta}\right] \simeq \sqrt{\frac{\beta}{2}}\,G_{k+\frac{1}{2}}(\eta), \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2 \tag{4-6}$$

and

$$\frac{\partial F_k(\eta,\beta)}{\partial\eta} \simeq \sqrt{\frac{\beta}{2}}\,\frac{\partial G_{k+\frac{1}{2}}(\eta)}{\partial\eta}, \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2. \tag{4-7}$$

4.2. The NonDegenerate Limit — $\eta < -30$

In the nondegenerate limit, we make use of the well-known (Chandrasekhar 1939) relations among the Fermi-Dirac integrals and the modified Bessel functions of the second kind K_I and K_{II} (we shall henceforth not explicitly write the adjective "modified"). The Bessel functions are defined by

$$K_I(\phi) = \int_0^\infty \cosh(t)\exp\left[-\phi\cosh(t)\right]dt \tag{4-8}$$

and

$$K_{II}(\phi) = \int_0^\infty \cosh(2t)\exp\left[-\phi\cosh(t)\right]dt, \tag{4-9}$$

where

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2} \tag{4-10}$$

is the hyperbolic cosine and

$$\phi = \frac{1}{\beta}. \tag{4-11}$$

Defining

$$\mathcal{B} = \frac{1}{\sqrt{2}} \exp\left(\eta + \phi\right), \tag{4-12}$$

the nondegenerate Fermi-Dirac functions are

$$F_{1/2}(\eta, \beta) \simeq \mathcal{B}\phi^{\frac{1}{2}} K_I(\phi), \tag{4-13a}$$

$$F_{3/2}(\eta, \beta) \simeq \mathcal{B}\phi^{\frac{3}{2}} \left(K_{II}(\phi) - K_I(\phi)\right), \tag{4-13b}$$

$$F_{5/2}(\eta, \beta) \simeq \mathcal{B}\phi^{\frac{5}{2}} \left[2K_I(\phi) + (3\beta - 2)\, K_{II}(\phi)\right], \tag{4-13c}$$

the Fermi-Dirac functions are the same as their η-derivatives,

$$\frac{\partial F_k(\eta, \beta)}{\partial \eta} \simeq F_k(\eta, \beta), \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2, \tag{4-14}$$

and the β-derivatives are

$$\frac{\partial F_{1/2}(\eta, \beta)}{\partial \beta} \simeq \mathcal{B}\phi^{\frac{3}{2}} \left[\phi\left(K_{II} - K_I\right) - \frac{3}{2}K_I\right], \tag{4-15a}$$

$$\frac{\partial F_{3/2}(\eta, \beta)}{\partial \beta} \simeq \mathcal{B}\phi^{\frac{5}{2}} \left[2\phi\left(K_I - K_{II}\right) + \frac{1}{2}\left(5K_I + K_{II}\right)\right], \tag{4-15b}$$

and

$$\frac{\partial F_{5/2}(\eta, \beta)}{\partial \beta} \simeq \mathcal{B}\phi^{\frac{5}{2}} \left[\frac{3}{2}K_{II} - 2\phi\left(2K_I + K_{II}\right) + 4\phi^2\left(K_{II} - K_I\right)\right]. \tag{4-15c}$$

We make use of a combination of expansions and tabulations to evaluate K_I and K_{II}.

4.2.1. Chebyshev Series Expansions

Accurate Chebyshev series expansions exist for the Bessel functions. Tooper (1969) gives expansions, valid for $\phi > 8$, for K_0 and K_I:

$$K_j(\phi) = \frac{e^{-\phi}}{\sqrt{\phi}} {\sum_{k=0}^{\infty}}' a_k^j\, T_k\left(\frac{16}{\phi} - 1\right) \quad j = 0, I, \tag{4-16}$$

where the $'$ in the summation means

$$\sum_{k=0}^{\infty}{}' a_k\,T_k \equiv \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\,T_k. \tag{4-17}$$

(Tooper's expressions [Tables 13 and 14] for the Chebyshev expansions of K_0 and K_I contain an extra factor of π.) The Chebyshev polynomials $T_k(x)$ need not be explicitly calculated if the series is evaluated recursively. Starting with

$$b^j_{N+1} = b^j_{N+2} = 0, \tag{4-18}$$

successive iterates are given by

$$b^j_k = 2\left(\frac{16}{\phi} - 1\right) b^j_{k+1} - b^j_{k+2} + a^j_k, \quad k = N, \cdots 0, \tag{4-19}$$

and finally the sum

$$\sum_{k=0}^{N}{}' a^j_k\,T_k\left(\frac{16}{\phi} - 1\right) = \frac{1}{2}\left(b^j_0 - b^j_2\right). \tag{4-20}$$

We use Tooper's coefficients from his Tables 13 ($j = 0$) and 14 ($j = 1$), for which $N = 14$, to obtain

$$K_0(\phi) = \frac{1}{2}\frac{e^{-\phi}}{\sqrt{\phi}}\left(b^0_0 - b^0_2\right), \quad K_I(\phi) = \frac{1}{2}\frac{e^{-\phi}}{\sqrt{\phi}}\left(b^I_0 - b^I_2\right). \tag{4-21}$$

The function K_{II} is given by

$$K_{II}(\phi) = K_0(\phi) + \frac{2}{\phi}K_I(\phi). \tag{4-22}$$

For $\phi \le 8$, Chebyshev expansions from Clenshaw (1962) are applicable:

$$K_0(\phi) = -\ln\left(\frac{\phi}{8}\right)\sum_{k=0}^{\infty}{}' a^{00}_{2k}\,T_{2k}\left(\frac{\phi}{8}\right) + \sum_{k=0}^{\infty}{}' a^{01}_{2k}\,T_{2k}\left(\frac{\phi}{8}\right) \tag{4-23a}$$

and

$$K_I(\phi) = -\ln\left(\frac{\phi}{8}\right)\frac{\phi}{8}\sum_{k=0}^{\infty}{}' a^{I0}_{2k}\,T_{2k}\left(\frac{\phi}{8}\right) + \frac{1}{\phi} - \frac{\phi}{8}\sum_{k=0}^{\infty}{}' a^{I1}_{2k}\,T_{2k}\left(\frac{\phi}{8}\right) \tag{4-23b}$$

Each of the series in (4-23b) is evaluated by recursion relations similar to those used in the evaluation of (4-16). Starting with (4-18) and using the Chebyshev coefficients a^j_{2k} tabulated by Clenshaw for $N = 17$, the recursion

$$b^j_k = 2\left[2\left(\frac{\phi}{8}\right)^2 - 1\right] b^j_{k+1} - b^j_{k+2} + a^j_{2k}, \quad k = N, \cdots 0, \quad j = 00, 01, I0, I1 \tag{4-24}$$

is applied until the sum

$$\sum_{k=0}^{N}{}' a^j_{2k}\, T_{2k}(x) = \frac{1}{2}(b^j_0 - b^j_2), \quad j = 00, 01, I0, I1. \tag{4-25}$$

4.2.2. NonDegenerate Small-β Expansions

In the nonrelativistic limit, $\beta \rightarrow 0$, a power series asymptotic expansion (Gradshteyn & Ryzhik 1980) in β is applicable. After expanding K_I and K_{II} and collecting terms,

$$F_{1/2}(\eta,\beta) = \frac{\partial F_{1/2}(\eta,\beta)}{\partial \eta} \simeq \frac{\sqrt{\pi}}{2}\exp(\eta)\left(1 + \frac{3}{8}\beta - \frac{15}{128}\beta^2 + \frac{105}{1024}\beta^3 - \frac{105}{1024}\beta^4\right), \tag{4-26a}$$

$$F_{3/2}(\eta,\beta) = \frac{\partial F_{3/2}(\eta,\beta)}{\partial \eta} \simeq \frac{3\sqrt{\pi}}{4}\exp(\eta)\left(1 + \frac{5}{8}\beta - \frac{35}{128}\beta^2 - \frac{2345}{16384}\beta^3\right), \tag{4-26b}$$

$$F_{5/2}(\eta,\beta) = \frac{\partial F_{5/2}(\eta,\beta)}{\partial \eta} \simeq \frac{15\sqrt{\pi}}{8}\exp(\eta)\left(1 + \frac{7}{8}\beta - \frac{539}{4090}\beta^2\right), \tag{4-26c}$$

$$\frac{\partial F_{1/2}(\eta,\beta)}{\partial \beta} \simeq \frac{3\sqrt{\pi}}{16}\exp(\eta)\left(1 - \frac{5}{8}\beta + \frac{105}{128}\beta^2 - \frac{35}{32}\beta^3\right), \tag{4-27a}$$

$$\frac{\partial F_{3/2}(\eta,\beta)}{\partial \beta} \simeq \frac{15\sqrt{\pi}}{32}\exp(\eta)\left(1 - \frac{7}{8}\beta - \frac{1407}{2048}\beta^2\right), \tag{4-27b}$$

and

$$\frac{\partial F_{5/2}(\eta,\beta)}{\partial \beta} \simeq \frac{105\sqrt{\pi}}{34}\exp(\eta)\left(1 - \frac{77}{256}\beta\right). \tag{4-27c}$$

The higher order terms in the expansion (4-27c) have cancelled, but since this expression is only used for $\beta < 10^{-3}$, the remaining terms give sufficient accuracy.

4.2.3. Extremely Relativistic and NonDegenerate

In the relativistic limit, which we employ for $\beta > 10^4$, $K_I(x) \simeq\ 1/x$, $K_{II}(x) \simeq 2/x^2$, and (4-13a) through (4-15c) become

$$F_k(\eta,\beta) = \frac{\partial F_k(\eta,\beta)}{\partial\eta} \simeq \left(k + \frac{1}{2}\right)! \exp\left(\eta + \frac{1}{\beta}\right)\sqrt{\frac{\beta}{2}}, \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2 \tag{4-28}$$

and

$$\frac{\partial F_k(\eta,\beta)}{\partial\beta} \simeq \left(\frac{1}{2\beta} - \frac{1}{\beta^2}\right) F_k(\eta,\beta), \qquad k = {}^1\!/_2,\ {}^3\!/_2,\ {}^5\!/_2. \tag{4-29}$$

4.2.4. Bessel Function Tabulation and Evaluation Methods

We have prepared, by numerical integration, tabulations of the Bessel functions on a grid of $-4 \le \log_{10}\phi \le 1.6$ with a spacing $\Delta\log_{10}\phi = 0.1$.

For $\phi > 1$, evaluations of derivatives by (4-15a), (4-15b), and especially (4-15c) involve cancellation between the K_I and K_{II} terms and require more precision than exists in our table. In addition, K_I and K_{II} decrease ever more rapidly with increasing ϕ; this sharp falloff causes loss of interpolation accuracy for $\phi > 1$ ($\beta < 1$). Greater accuracy is obtained by use of high ($6^{\rm th}$) order interpolation in K_I and K_{II} and the second expression in equations 4-15a, 4-15b, and 4-15c, which result in better cancellation, but the tables should only be used above $\beta = \beta_t$. Between β_t and the smaller β_c, the Chebyshev expansions should be used. For $\beta < \beta_c$, the small-β expansions are more accurate than the Chebyshev series, for which the accuracy suffers as β decreases.

For $6^{\rm th}$ order interpolation, the tables, rather than the Chebyshev series are used for $\beta > \beta_t = 10^{0.3}$. Across this boundary, the Fermi-Dirac functions and derivatives match to 1 part in 5×10^5 or less. For less accurate, lower order interpolations, larger values of β_t may be appropriate (unless lower order interpolation is used for the sake of

reducing computer time). The computer time taken to evaluate a set of functions with the Chebyshev expansions is 1.09 times greater than evaluating the same set with 6^{th} order table interpolation.

When the β-derivatives are required, the small-β expansions should be used for $\beta < \beta_c = 10^{-2.8}$; for $\frac{\partial F_{5/2}(\eta,\beta)}{\partial\beta}$, the loss of accuracy with increasing or decreasing β is steep away from the respective side of the switching point. When the derivatives are not required, $\log\beta_c = -2.0$ is recommended. Across the $\beta_c = 10^{-2.8}$ boundary, the functions, the η-derivatives, and $\frac{\partial F_{1/2}(\eta,\beta)}{\partial\beta}$ match to a relative difference of 10^{-6} or better, $\frac{\partial F_{3/2}(\eta,\beta)}{\partial\beta}$ to 10^{-5}, and $\frac{\partial F_{5/2}(\eta,\beta)}{\partial\beta}$ to 0.003. The evaluation of the same set of functions with the Chebyshev expansion requires 3.3 times as much computer time as with the small-β expansions.

An alternative method of calculating the nondegenerate quantities, which we do not employ, makes use the values of the Fermi-Dirac functions and derivatives in the central table. The η dependence is given by equation 4-12, so that

$$X(\eta,\beta) = \exp(\eta+30)\,X(-30,\beta),$$

where X is any F_k or derivative.

5. The Tables: Creation, Interpolation, and Accuracy

5.1. Numerical Integration

All integrals considered here were numerically evaluated using a 7-point adaptive Newton–Cotes quadrature rule (implemented in the routine QNC79 from the SLATEC software library). For the Fermi-Dirac and Fermi integrals, $\eta + 100$ (rather than ∞) was used for the upper limit of the numerical integration; the lower limit was $\max(0, \eta - 100)$, whereas for the Bessel functions the integrations ran from 0 to $6 - \ln(\phi)$. The integration

routine evaluates each integral until a desired accuracy is achieved. The accuracy is expressed as a tolerance $\mathcal{E}$, where the result of the numerical integration I does not deviate more than $\mathcal{E}I$ from the true answer. We used $\mathcal{E} = 10^{-12}$. A comparison of integrations made on 2 platforms—a Silicon Graphics Indigo2 with an R4400 cpu chip running IRIX 5.3 and an Apollo DN4000 running DOMAIN/OS 10.3—agreed to within a relative difference of 1.2×10^{-7}. This comparison suggests the accuracy of our integrations is not better than 1 part in 10^{-7}. The values in the table are the Silicon Graphics integrations.

5.2. The Tables

Three tables of integrals accompany the electronic version of this paper. The file `fdints.tab` contains the Fermi-Dirac integrals and the Fermi integrals. The file `dfdints.tab` contains the derivatives of the Fermi-Dirac and of the Fermi integrals. The Bessel functions are found in `bessel.tab`. The file `fermi7h.tab` contains the Fermi integrals of order $^7/_2$. These tables are read by the FORTRAN subroutine `calcdfi`, which is also provided along with the tables; the exact format of the tables can be found by examining the relevant `READ` statements in the subroutine. The functions and the derivatives are maintained as separate files to facilitate implementations where only the function values are required; subroutine `calcfi`, also supplied, is a variant of `calcdfi` with the derivatives stripped out.

Subroutine `calcdfi` reads the data from the binary (or unformatted) files `fdints.unf` (which contains the Bessel data) and `dfdints.f` is they both exist; if they do not exist, the formatted files `fdints.tab`, $\cdots$ are read, and the logarithm of the data are written to the binary files. The binary files are preferred because the formatted files are 5 times as large and take 35 times longer to load. All data is held in memory as (natural) logarithms.

The main table of the Fermi-Dirac functions and their integrals for $-30 \leq \eta \leq 70$ and $-6 \leq \log\beta \leq 4$, the same ranges as used in the tabulations in Appendix A.2 of Cox & Giuli (1968). The table uses a finer grid for small values of $|\eta|$ and $|\beta|$. The table contains 21 β values corresponding to the integral and half-integral values of $-6 \leq \log(\beta) \leq 4$. The 47 points in the η grid are concentrated towards $\eta = 0$. The spacing is 0.1 for $-1 \leq \eta \leq 1$, 1 for $-5 \leq \eta \leq -1$ and $1 \leq \eta \leq 5$, and 5 for $-30 \leq \eta \leq -5$ and $5 \leq \eta \leq 70$. The η grid for the Fermi integrals and derivatives consists of 101 values of integral $-30 \leq \eta \leq 70$. The ϕ grid for the Bessel functions contains 59 values with a spacing $\Delta \log_{10}\phi = 0.1$.

5.3. Interpolation

All interpolation is made with the logarithms of the integrals and derivatives as functions of $\log\beta$ (or $\log\phi$) and η (except as noted in the discussion of second order interpolation in the 2-dimensional table). Use of the direct values, rather than the logarithms, is much less accurate for all interpolation orders.

5.3.1. Second Order Interpolation

Second order interpolation for the 1-dimensional functions is a simple linear interpolation between the values on bracketing grid points. For the central 2-dimensional tables, the interpolation is logarithmic in the η direction

$$f(\beta) = \exp\left\{(1-d)\ln[f(\eta_a,\beta)] + d\ln[f(\eta_b,\beta)]\right\} \tag{5-1}$$

where d is the interpolation coefficient and $\eta_{(a,b)}$ are points in the table. The $\frac{2}{3}$ power is used for the interpolation in the β direction,

$$f = \left[(1-d)f(\beta_a)^{\frac{2}{3}} + df(\beta_b)^{\frac{2}{3}}\right]^{\frac{3}{2}} \tag{5-2}$$

where d and $\beta_{(a,b)}$ have meanings similar to the above.

5.3.2. Higher Order Polynomial Interpolation

Our polynomial interpolation is based on the subroutines `POLINT` and `POLIN2` from the *Numerical Recipes* book (Press, et al. 1986). We examined the accuracy and speed of the interpolation for several polynomial orders. We found 6^{th} order to be most accurate, but with a substantial penalty in execution speed. Orders 2 through 6 are available in the subroutine we provide.

The tables below give the accuracy and execution time for various orders and functions. The accuracy is given in terms of the relative error $\max(f/f\prime, f\prime/f)$, where f is truth, as given by a numerical integration, and $f\prime$ is the value from interpolation. Execution times are normalized to 1 for 2^{nd} order. The timings were made with uniform samplings over the respective tables.

5.3.3. The Two Dimensional Tables

Table 1 gives the maximum relative error for the Fermi-Dirac integrals, along with the execution times, and Table 2 gives the maximum relative errors for the derivatives. The accuracy is based on comparisons with numerical integrations made at $\eta = 0.6\eta_i + 0.4\eta_{i+1}$ and $\log\beta = 0.6\log\beta_j + 0.4\log\beta_{j+1}$ for each (i, j) cell in the table, a similar set of integrations with $0.4 \leftrightarrow 0.6$ for every other (i, j) cell, and a sampling of cells with a set of 10 integrations crossing the cell in some direction.

The order in which the 1-dimensional interpolations were performed (ie. η direction first or β direction first) made no difference in the accuracy.

EDITOR: PLACE TABLE 1 HERE.

EDITOR: PLACE TABLE 2 HERE.

5.3.4. Fermi Integrals

Table 3 gives the maximum error in the Fermi integrals and their derivatives for a range of ± 0.5 in η centered on the value listed. There is variation of up to 100% among the relative errors for the individual functions and derivatives. The maximum relative errors for the 6 functions and the 6 derivatives are the same to within 1%. The relative execution times for the interpolations are given in Table 4.

EDITOR: PLACE TABLE 3 HERE.

EDITOR: PLACE TABLE 4 HERE.

5.3.5. Bessel Function Interpolation

Table 5 lists the maximum relative errors in the Bessel functions for several ranges of ϕ. For each ϕ range in each order, the maximum relative errors for K_I and K_{II} were comparable. The larger of the two is listed in the table. The error becomes increasingly worse as ϕ increases and the Bessel functions are dominated by the factor $\exp -\phi$.

EDITOR: PLACE TABLE 5 HERE.

5.3.6. Accuracy at Treatment Boundaries

Different methods are used to evaluate the Fermi-Dirac integrals and their derivatives for different ranges of η and β. The closeness of the evaluations on opposite sides of a treatment boundary is given in Table 6. The quantity tabulated is the largest value of $\max(f_r/f_l, f_l/f_r)$ along a boundary. For an η boundary, $f_{(r,l)} = f(\eta \pm 0.000001)$, and for a β boundary $f_{(r,l)} = f(\log\beta \pm 0.000001)$. The results are most excellent, especially for 6$^{\text{th}}$ order interpolation.

The β-derivatives lose all accuracy near $\log\beta = -1.6$. Accordingly, we also list the closeness along $\eta = -30$ excluding a range near $\log\beta = -1.6$.

EDITOR: PLACE TABLE 6 HERE.

We are grateful to Onno Pols for helping us correctly implement the Eggleton, Faulkner, & Flannery formulae, and to Jim Lattimer for discussions on various approximations.

Table 1. Main Table Interpolation.

		Maximum Relative Error		
N	time	$F_{1/2}(\eta,\beta)$	$F_{3/2}(\eta,\beta)$	$F_{5/2}(\eta,\beta)$
2	1.0	1.082	1.118	1.143
3	1.8	1.572	1.646	1.628
4	2.9	1.023	1.023	1.021
5	4.6	1.027	1.016	1.009
6	7.1	1.007	1.003	1.004
7	10.	1.005	1.010	1.012
8	14.	1.013	1.007	1.005
9	19.	1.096	1.050	1.026

Table 2. Main Table Derivative Interpolation.

	Maximum Relative Error					
N	$\frac{\partial F_{1/2}(\eta,\beta)}{\partial\eta}$	$\frac{\partial F_{3/2}(\eta,\beta)}{\partial\eta}$	$\frac{\partial F_{5/2}(\eta,\beta)}{\partial\eta}$	$\frac{\partial F_{1/2}(\eta,\beta)}{\partial\beta}$	$\frac{\partial F_{3/2}(\eta,\beta)}{\partial\beta}$	$\frac{\partial F_{5/2}(\eta,\beta)}{\partial\beta}$
2	1.034	1.082	1.118	1.132	1.159	1.177
3	1.391	1.572	1.646	1.573	1.646	1.628
4	1.019	1.023	1.024	1.024	1.023	1.018
5	1.032	1.027	1.016	1.022	1.010	1.010
6	1.011	1.007	1.003	1.005	1.004	1.005
7	1.019	1.006	1.010	1.007	1.012	1.013
8	1.026	1.013	1.007	1.010	1.006	1.005
9	1.187	1.096	1.050	1.069	1.036	1.019

Table 3. Fermi Function Interpolation Relative Errors.

	Interpolation Order				
η	2	3	4	5	6
−29.5	1.0000006	1.0000006	1.0000006	1.0000006	1.0000006
−19.5	1.0000006	1.0000006	1.0000006	1.0000006	1.0000006
−14.5	1.0000005	1.0000023	1.0000007	1.0000008	1.0000006
−9.5	1.0000068	1.0000182	1.0000011	1.0000006	1.0000005
−4.5	1.0009817	1.0025651	1.0001572	1.0000744	1.0000216
−2.5	1.0063697	1.0168104	1.0008456	1.0003333	1.0000158
−1.5	1.0137371	1.0364101	1.0011945	1.0002281	1.0002464
−0.5	1.0225276	1.0566500	1.0005099	1.0009853	1.0002920
0.5	1.0247931	1.0412589	1.0026541	1.0018509	1.0004923
1.5	1.0187399	1.0212393	1.0022867	1.0005230	1.0003849
2.5	1.0132760	1.0471996	1.0007911	1.0016337	1.0002560
4.5	1.0091014	1.0247539	1.0006357	1.0002153	1.0000434
9.5	1.0041356	1.0051802	1.0000569	1.0000178	1.0000018
14.5	1.0020823	1.0019885	1.0000169	1.0000033	1.0000004
19.5	1.0012197	1.0008866	1.0000060	1.0000016	1.0000005
24.5	1.0007939	1.0004615	1.0000026	1.0000006	1.0000005
29.5	1.0005556	1.0002668	1.0000014	1.0000004	1.0000004
34.5	1.0004097	1.0001662	1.0000010	1.0000005	1.0000005
39.5	1.0003149	1.0001134	1.0000009	1.0000006	1.0000005
44.5	1.0002490	1.0000782	1.0000005	1.0000005	1.0000005
49.5	1.0002017	1.0000569	1.0000004	1.0000007	1.0000005
59.5	1.0001400	1.0000308	1.0000005	1.0000006	1.0000005
69.5	1.0001032	1.0000216	1.0000021	1.0000034	1.0000067

Table 4. Interpolation Times.

	Relative Time	
N	G, $\partial G/\partial\eta$	K_I, K_{II}
2	1.0	1.0
3	7.6	4.9
4	11.	7.1
5	16.	9.8
6	22.	13.

Table 5. Bessel Function Interpolation Relative Errors.

		Interpolation Order				
$\log \phi_a$	$\log \phi_b$	2	3	4	5	6
-4.0	-2.0	1.000066	1.000139	1.000042	1.000047	1.000042
-2.0	0.0	1.005176	1.008385	1.000117	1.000050	1.000041
0.0	0.5	1.018382	1.028126	1.000306	1.000092	1.000059
0.5	1.0	1.060710	1.090631	1.000918	1.000201	1.000088
1.0	1.5	1.205830	1.311856	1.002875	1.000673	1.000389

Table 6. Difference at Treatment Boundaries.

		Interpolation Order				
Boundary	Function	2	3	4	5	6
$\eta = 70$	F	1.0007	1.0026	1.0022	1.0582	1.0309
$\eta = 70$	$\partial F/\partial \eta$	1.0008	1.0024	1.0030	1.0632	1.0308
$\eta = 70$	$\partial F/\partial \beta$	1.0011	1.0023	1.0038	1.0611	1.0279
$\eta = -30$	F	1.0027[a]	1.0022	1.0029	1.0473	1.0303
$\eta = -30$	$\partial F/\partial \eta$	1.0027[a]	1.0022	1.0029	1.0473	1.0303
$\eta = -30$	$\partial F/\partial \beta$	1.0021[a]	1.0020	1.0024	1.0445	1.0279
$\log \beta = 4$	F	1.0072	1.0221	1.0231	1.5966	1.1481
$\log \beta = 4$	$\partial F/\partial \eta$	1.0081	1.0301	1.0231	1.5967	1.1270
$\log \beta = 4$	$\partial F/\partial \beta$	1.0064	1.0207	1.0225	1.5971	1.1481
$\log \beta = -6$	F	1.0071	1.0262	1.0227	1.5976	1.1386
$\log \beta = -6$	$\partial F/\partial \eta$	1.0101	1.0312	1.0227	1.5752	1.1132
$\log \beta = -6$	$\partial F/\partial \beta$	1.0042	1.0134	1.0213	1.5864	1.1258

[a]The difference is 1.0007 when the range $3.7 \leq \log \beta \leq 4.0$ is excluded

REFERENCES

Antia, H. M. 1993, ApJS, 84, 101.

Beaudet, G., & Tassoul, M. 1971, A&A, 13, 209.

Blinnikov, S. I., & Rudzskii, M. A. 1988, Astrofizika, 29, 385.

Bludman, S. A. & Van Riper, K. A. 1977, ApJ, 212, 859.

Chandrasekhar, S. 1939, An Introduction to the Study of Stellar Structure (Chicago: University of Chicago Press).

Clenshaw, C. W. 1962, Chebyshev Series for Mathematical Functions, (London: Her Majesty's Stationery Office), 510 N279m V.5.

Cox, J. P., & Giuli, J. P. 1968, Principles of Stellar Structure, Vol 2 New York: Gordon & Breach, chap. 24.

Divine, N. 1965, ApJ, 142, 1652.

Eggleton, P. P., Faulkner, J., & Flannery, B. P. 1973, å, 23, 325.

Gradshteyn, I. S. & Ryzhik, I. M. 1980, Table of Integrals, Series, and Products, New York: Academic Press.

Guess, A. W. 1966, Adv. Astr. Ap., 4, 153.

Kippenhahn, R. & Thomas, H.–C. 1964, Z. F. Ap., 60, 19.

Kippenhahn, R., Weigert, A., & Hofmeister, E. 1967, Methods in Computational Physics, 7, 129.

Nadyozhin, D. K. 1974, Nauchnaya Informatsiya Astrosovieta Akad. Nauk SSSR (Sci. Inform. Astron. Council of the USSR), No. 32, 3.

Pols, O. R., Tout, C. A., Eggleton, P. P., & Han, Z. 1995, MNRAS, 274, 964.

Press, W. H., Flannery, B. P., Teukolsky, S. A., & Vettering, W., T 1986, Numerical Recipes, Cambridge: Cambridge, Chapter 3.

Tooper, R. F. 1969, ApJ, 156, 1075.

Urpin, V. A. 1981, Astrofizika, 18, 133.

This manuscript was prepared with the AAS LATEX macros v4.0.